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Theory and Methodology

Dynamic firm behavior within an uncertain environment

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Abstract: In this paper we study the impact of an uncertain environment on the optimal dynamic investment policies of a value maximizing firm. We present a model in which risk-averse behavior of the shareholders is incorporated. We derive the policies that can be optimal for the firm and present solutions under different scenarios. After incorporating a dynamic version of the Capital Asset Pricing Model we can derive a new formula for the shareholders' time preference rate, which consists of the riskless interest rate and a risk premium.

Keywords: Investment, finance, dynamic programming

1. Introduction

Application of stochastic dynamic optimization methods to economic problems of the firm received a lot of attention in the recent literature. Here, we can think of the valuation of investment projects (e.g. [13]), the determination of the optimal investment portfolio (e.g. [8]) and quality control (see [16]). In this paper we study the optimal investment/dividend policy within the context of a dynamic model of the firm. In a deterministic context the dynamic theory of the firm has been a fruitful area for interesting scientific contributions, e.g. [6,9,14]. The purpose of this paper is to extend these contributions by adding another dimension: uncertainty.

The first stochastic dynamic model of the firm was designed by Bensoussan and Lesourne [3]. A main difference between this model and the deterministic models mentioned before is the presence of cash. Within an uncertain environment earnings can fall below the expenses level, so net cash outflows may occur. Therefore a certain amount of cash is needed to meet the firm's obligations during such periods. In deterministic models there is no reason for holding cash, but in a stochastic model of the firm it makes sense to analyse the firm's cash decision. Another difference with the deterministic models is that the planning horizon is endogeneously determined, namely as the point of time where the amount of cash becomes negative, i.e. when the firm goes bankrupt.

One of the results of the static theory of the Capital Asset Pricing Model is that the discount rate depends on the amount of risk the firm has to deal with (see [2, p. 146]). Like in deterministic models of

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the firm, Bensoussan and Lesourne assume the shareholders' time preference rate to be constant. But due to the uncertain environment, the firm has to deal with risk in this model. Therefore it seems interesting to incorporate a dynamic version of the CAPM in the stochastic dynamic model of the firm.

In Section 2 we present our dynamic model of the firm in which the Bensoussan and Lesourne model is extended by changing the objective from dividend maximization into the maximization of the utility stream of dividend. Here, we can apply the CAPM approach, which is only valid under the assumption of risk-averse investors. In Section 3 the model is solved while in Section 4 the Intertemporal Capital Asset Pricing Model, invented by Merton ([11,12]), is incorporated. In Section 5 we summarize our findings.

2. Model formulation

In this section the stochastic dynamic model of the firm is presented. The stochastic part of the model is incorporated in the earnings function, which can be expressed as

$$R(K) = S(K)(1 + \sigma W) \tag{1}$$

in which

K = capital good stock,
$R(K)$ = earnings function,
$S(K)$ = usual deterministic earnings function (see e.g. [9]), $\mathrm{d}S/\mathrm{d}K > 0$, $S(0) = 0$, $\mathrm{d}S/\mathrm{d}K\,|_{K=0} > i$, where i = the shareholders' time preference rate,
W = Gaussian stochastic variable, $E(W) = 0$, $\mathrm{Var}(W) = 1$, $W(t)$ and $W(\hat{t})$ are independently distributed if $t \neq \hat{t}$,
σ = a constant.

To apply dynamic programming we rewrite (1) into an Itô stochastic differential equation (see e.g. [1]). Before we do this, first notice that $W(t)\,\mathrm{d}t$ can be formally expressed as $\mathrm{d}B(t)$, where $B(t)$ is a standard Wiener process (see [15, p. 296]). If we multiply (1) by $\mathrm{d}t$ we obtain

$$R(K)\,\mathrm{d}t = S(K)\,\mathrm{d}t + \sigma S(K)\,\mathrm{d}B. \tag{2}$$

We now formulate the model in symbols; afterwards the interpretation of the model will be given:

$$\underset{\dot{K},\,D}{\text{maximize}} \quad E\left(\int_0^T V(D)\,\mathrm{e}^{-it}\,\mathrm{d}t\right) \tag{3}$$

$$\text{subject to} \quad \mathrm{d}K = \dot{K}\,\mathrm{d}t, \tag{4}$$

$$\mathrm{d}M = \left(S(K) - \dot{K} - D\right)\mathrm{d}t + \sigma S(K)\,\mathrm{d}B, \tag{5}$$

$$D \geqslant 0, \qquad \dot{K} \geqslant 0, \qquad S(K) - \dot{K} - D \geqslant 0, \tag{6, 7, 8}$$

$$K(0) = K_0 > 0, \qquad M(0) = M_0 > 0, \tag{9, 10}$$

in which

D = dividend,
M = cash balance,
t = time,
$V(D)$ = utility function of the shareholders, $\mathrm{d}V/\mathrm{d}D > 0$, $\mathrm{d}^2V/\mathrm{d}D^2 < 0$, $V(0) = 0$,
T = horizon date,
i = shareholders' time preference rate.

The firm behaves as if it maximizes the shareholders' value of the firm. This value is expressed as the mathematical expectation of the discounted utility stream of dividends where the utility function is concave. The firm is bankrupt as soon as M becomes negative. In this way the horizon date T is endogeneously determined and can be expressed as follows:

$$T = \inf\{t \mid M(t) \leqslant 0\}.$$

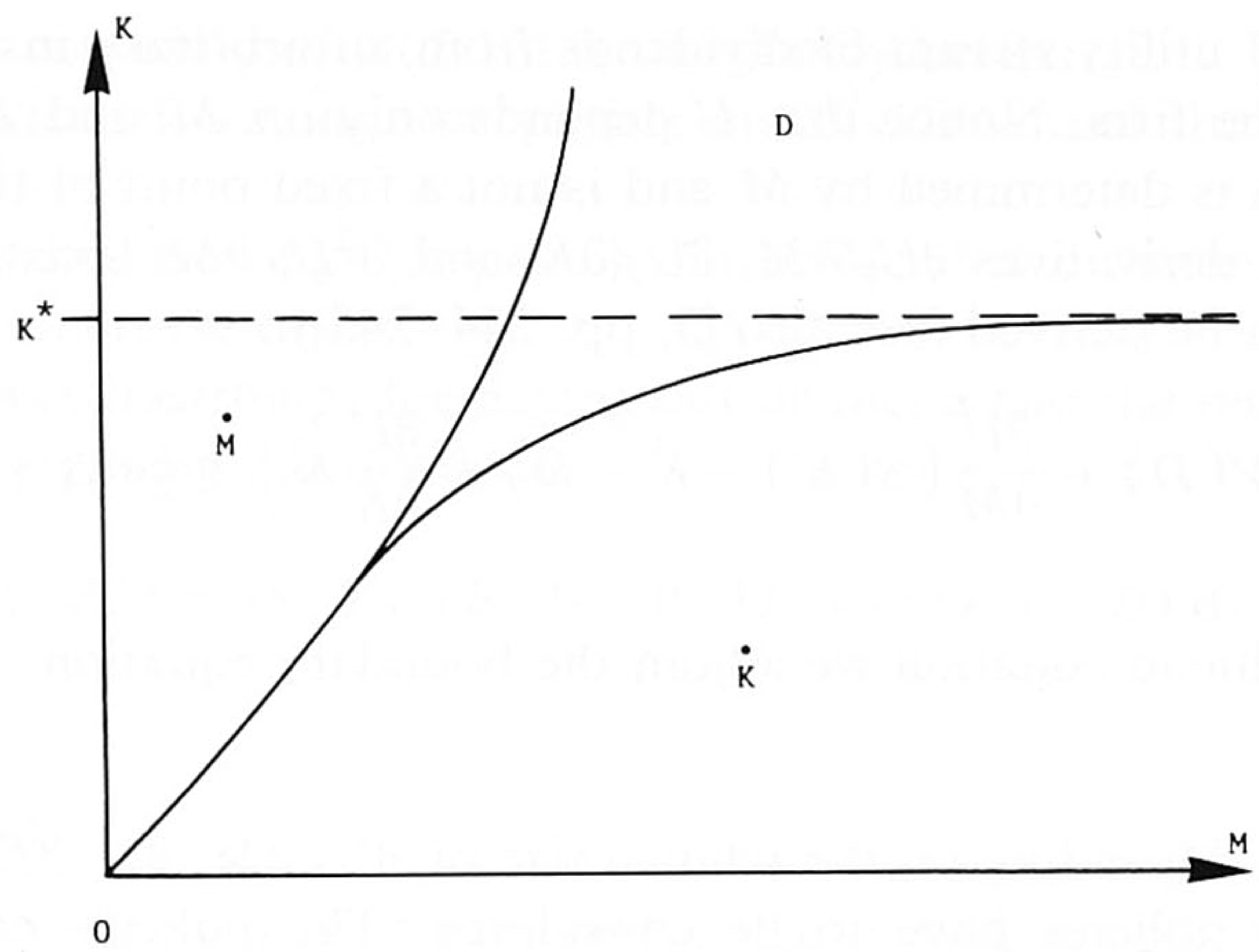

Figure 1. The most realistic part of the optimal solution of Bensoussan and Lesourne's model

We further suppose that there are no depreciations (4). Due to (2) and (5), we derive that the firm can spend its earnings in three directions: increase the cash balance, invest in capital goods or pay dividend. In order to get the pay off (3) to be well defined (see [3, p. 240]), the controls have to be bounded by deterministic constants. This is the case in our model, because dividends are bounded by a rational lower bound (6), and further it is assumed that investments are irreversible (7), and that, at any time, the firm does not spend more money on investment and dividend than the expected earnings (8). (8) may be considered as a 'wisdom rule' according to which it would be unreasonable not to base investment and dividend distributions on the expected earnings. Debt is not included in the model, because we want to focus primarily on the cash management problem. Bensoussan and Lesourne [4] have carried out some numerical experiments in a stochastic model that includes the possibility of borrowing.

If we should change the objective from maximizing a utility stream of dividends into dividend maximization:

$$\int_0^T D\, \mathrm{e}^{-it}\, \mathrm{d}t, \tag{11}$$

we obtain the model designed by Bensoussan and Lesourne [3]. In [7] this model is extensively treated and the optimal solution is improved. The most realistic part of this solution is presented in Figure 1.

Figure 1 shows that depending on the level of cash and capital goods the firm carries out one of the following policies:

$\dot{M}$-policy: The firm keeps its cash if the amount of equipment is high enough while the cash situation is poor.

$\dot{K}$-policy: The firm invests if the amount of equipment is low, while there is plenty of cash to limit the risk of bankruptcy.

D-policy: The firm distributes dividends if M and K are such that the marginal profitability of investment is too small to justify additional growth and the amount of cash available high enough to guarantee a sufficiently safe situation.

In the remainder of this paper we shall use the objective described by (3).

3. Solution

First, we introduce

$$U(M(t), K(t)) = \max_{\substack{\dot{K},\, D \geqslant 0 \\ \dot{K}+D \leqslant S(K)}} E\left(\int_t^T V(D)\, \mathrm{e}^{-i(s-t)}\, \mathrm{d}s\right). \tag{12}$$

U is the expected discounted utility stream of dividends from an arbitrary instant $t \in [0, T]$ and can be interpreted as the value of the firm. Notice that U depends only on M and K, and not explicitly on t, because the planning horizon is determined by M and is not a fixed point of time.

Assuming that the partial derivatives $\partial U/\partial M$, $\partial U/\partial K$ and $\partial^2 U/\partial M^2$ exist, the following Hamilton–Jacobi–Bellman equation can be derived (see also [3, pp. 244–245]):

$$iU = \max_{\substack{\dot{K},\, D \geqslant 0 \\ \dot{K} + D \leqslant S(K)}} \left\{ V(D) + \frac{\partial U}{\partial M}\left(S(K) - \dot{K} - D\right) + \frac{\partial U}{\partial K}\dot{K} \right\} + \tfrac{1}{2}\sigma^2 S^2(K)\frac{\partial^2 U}{\partial M^2}. \tag{13}$$

To the Hamilton–Jacobi–Bellman equation we adjoin the boundary equation

$$U(0, K) = 0. \tag{14}$$

In Appendix 1 we show that, depending on the relative size of $\partial U/\partial M$, $\partial U/\partial K$ and $\mathrm{d}V/\mathrm{d}D$ for different values of D, five candidate policies have to be considered. The policies can be easily economically interpreted since:

$\partial U/\partial K =$ the marginal increase of the value of the firm due to an additional investment of one dollar,
$\partial U/\partial M =$ the marginal increase of the value of the firm due to one extra dollar kept in cash,
$\mathrm{d}V/\mathrm{d}D =$ the marginal increase of the value of the firm due to an additional dollar used to distribute dividends.

The five optimal policies are the following:

Investment policy: $\mathrm{d}M = \sigma S(K)\,\mathrm{d}B$, $D = 0$, $\mathrm{d}K = S(K)\,\mathrm{d}t$;
optimal if:

$$\frac{\partial U}{\partial K} \geqslant \max\left(\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}, \frac{\partial U}{\partial K} \right). \tag{15}$$

Thus for this policy it is marginally better:
- to invest than to pay out dividend;
- to invest than to increase cash.

Cash policy: $\mathrm{d}M = S(K)\,\mathrm{d}t + \sigma S(K)\,\mathrm{d}B$, $D = 0$, $\mathrm{d}K = 0$;
optimal if:

$$\frac{\partial U}{\partial M} \geqslant \max\left(\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}, \frac{\partial U}{\partial K} \right). \tag{16}$$

Due to (16) we can conclude that for this policy it is marginally better:
- to increase cash than to pay out dividend;
- to increase cash than to invest.

Dividend policy: $\mathrm{d}M = \sigma S(K)\,\mathrm{d}B$, $D = S(K)$, $\mathrm{d}K = 0$;
optimal if:

$$\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)} \geqslant \max\left(\frac{\partial U}{\partial K}, \frac{\partial U}{\partial M} \right). \tag{17}$$

For this policy it is marginally better:
- to pay out dividend than to invest;
- to pay out dividend than to increase cash.

Cash / dividend policy: $\mathrm{d}M = (S(K) - D)\,\mathrm{d}t + \sigma S(K)\,\mathrm{d}B$, $D \geqslant 0$, $\mathrm{d}K = 0$;
optimal if:

$$\frac{\mathrm{d}V}{\mathrm{d}D} = \frac{\partial U}{\partial M} \geqslant \frac{\partial U}{\partial K}. \tag{18}$$

Due to (18) and the strict concavity of $V(D)$ it is marginally better:
- to use a part of the expected earnings for paying out dividend and the rest to increase cash, than to invest;
- to use a part of the expected earnings for paying out dividend and the rest to increase cash, than to use all expected earnings to increase cash;
- to use a part of the expected earnings for paying out dividend and the rest to increase cash, than to use all expected earnings for paying out dividend.

Investment / dividend policy: $\mathrm{d}M = \sigma S(K)\,\mathrm{d}B$, $D \geqslant 0$, $\mathrm{d}K = (S(K) - D)\,\mathrm{d}t$; optimal if:

$$\frac{\mathrm{d}V}{\mathrm{d}D} = \frac{\partial U}{\partial K} \geqslant \frac{\partial U}{\partial M}. \tag{19}$$

From (19) and the strict concavity of $V(D)$ we derive that it is marginally better:
- to use a part of the expected earnings for paying out dividend and the rest to invest, than to increase cash;
- to use a part of the expected earnings for paying out dividend and the rest to invest, than to use all expected earnings for investment;
- to use a part of the expected earnings for paying out dividend and the rest to invest, than to use all expected earnings for paying out dividend.

After we have established the five policies that can be optimal, we now derive at what level of M and K which of these policies will be carried out. To do so, we divide the M–K plane in five different regions, each of them corresponding to one of the five optimal policies. In this way we get the following regions: investment-region, cash-region, dividend-region, cash/dividend-region and investment/dividend-region. As additional assumptions we require that also the partial derivatives $\partial^2 U/\partial K^2$ and $\partial^2 U/\partial M\,\partial K$ exist.

In contrast to the model of Bensoussan and Lesourne [3] (where dividends are maximized, see Figure 1), in the present model (where utility of dividends is maximized) the boundary between the cash-region and the dividend-region does not exist for K positive. This is, because in the cash-region it holds that $\partial U/\partial M \geqslant (\mathrm{d}V/\mathrm{d}D)|_{D=0}$ and in the dividend-region $\partial U/\partial M$ must be less or equal to $(\mathrm{d}V/\mathrm{d}D)|_{D=S(K)}$. Due to the strict concavity of $V(D)$ we can conclude that for K positive $(\mathrm{d}V/\mathrm{d}D)|_{D=0} > (\mathrm{d}V/\mathrm{d}D)|_{D=S(K)}$, so in the M–K plane the cash-region and the dividend-region have no points in common. Therefore, there will always be a cash/dividend-region between the cash- and the dividend-region. The same reasoning can be applied to argue that the investment/dividend-region must always be situated between the investment-region and the dividend-region.

In order to be able to construct the solution, we state the following propositions, that show where the boundaries between the regions are situated if M goes to infinity.

Proposition 1. *On the boundary between the cash-region and the cash/dividend-region and on the boundary between the cash/dividend-region and the dividend-region it holds that also K must have an infinite value if M goes to infinity.*

Proof. See Appendix. □

Proposition 2. *If M goes to infinity, the boundary between the investment/dividend-region and the dividend-region, and between the investment-region and the investment/dividend-region, are both situated on the level K^*, for which $(\mathrm{d}S/\mathrm{d}K)|_{K=K^*} = i$.*

Proof. This proof follows the same steps as the proof of Proposition 1 and is therefore omitted. □

About the (non) existence of the intersection points of the boundaries we can establish the following:
- For K positive the boundaries between cash and cash/dividend $(\partial U/\partial M = (\mathrm{d}V/\mathrm{d}D)|_{D=0})$ and

between cash/dividend and dividend $(\partial U/\partial M = (\mathrm{d}V/\mathrm{d}D)|_{D=S(K)})$ do not intersect, because at one point $\partial U/\partial M$ cannot have two different values.

- Following the same reasoning for K positive, we can demonstrate that the boundaries between investment and investment/dividend $(\partial U/\partial K = (\mathrm{d}V/\mathrm{d}D)|_{D=0})$ and between investment/dividend and dividend $(\partial U/\partial K = (\mathrm{d}V/\mathrm{d}D)|_{D=S(K)})$ do not intersect.
- For K positive, the boundaries between cash and cash/dividend $(\partial U/\partial M = (\mathrm{d}V/\mathrm{d}D)|_{D=0})$ and between investment/dividend and dividend $(\partial U/\partial K = (\mathrm{d}V/\mathrm{d}D)|_{D=S(K)})$ do not intersect, because it is never optimal to invest at a point in the direct neighborhood of the intersection point.
- Following the same reasoning we can argue that the boundaries between cash/dividend and dividend $(\partial U/\partial M = (\mathrm{d}V/\mathrm{d}D)|_{D=S(K)})$ and between investment and investment/dividend $(\partial U/\partial K = (\mathrm{d}V/\mathrm{d}D)|_{D=0})$ do not intersect.

Due to the complexity of the model under consideration we were not able to characterize the solution in so much detail as can be done for the model with dividend maximization. But if we start with some reasonable assumptions we can determine the optimal policies for the firm depending on the different levels of M and K. As motivation for these assumptions we use the economic interpretation of Figure 1. Thus we take the solution of the model with dividend maximization as a starting point for deriving the optimal solution of the present model. Briefly stated, the assumptions are:

(a) The firm distributes dividends if M and K are sufficiently high.
(b) The firm keeps its cash if K is high enough but the cash situation is poor.
(c) The firm invests if the amount of cash is such that the risk of bankruptcy is limited, while the amount of equipment is low.

The proof, that shows that there are reasonable choices of the parameters and of the functions $S(K)$ and $V(D)$ such that the assumptions (a), (b) and (c) are satisfied, can be found in Appendix B. By using the above derived properties of the boundaries we construct the optimal solution, which is presented in Figure 2.

Like in the model of Bensoussan and Lesourne [3] (see Figure 1), in the model at hand it is not optimal to invest if K is greater than K^*. The reason is that, due to the concavity of $S(K)$, the expected marginal earnings $(\mathrm{d}S/\mathrm{d}K)$ fall below the return the shareholders demand $(=i)$. This feature also plays an important role in the solutions of deterministic models (see e.g. [9]).

In comparison with the solution represented by Figure 1, the present solution contains two more regions in which a mixed cash/dividend policy $(\dot{M}/D)$ and a mixed investment dividend policy $(\dot{K}/D)$ will be carried out, respectively. Concerning the cash/dividend-region, on its boundary with the cash-region $(\dot{M})$

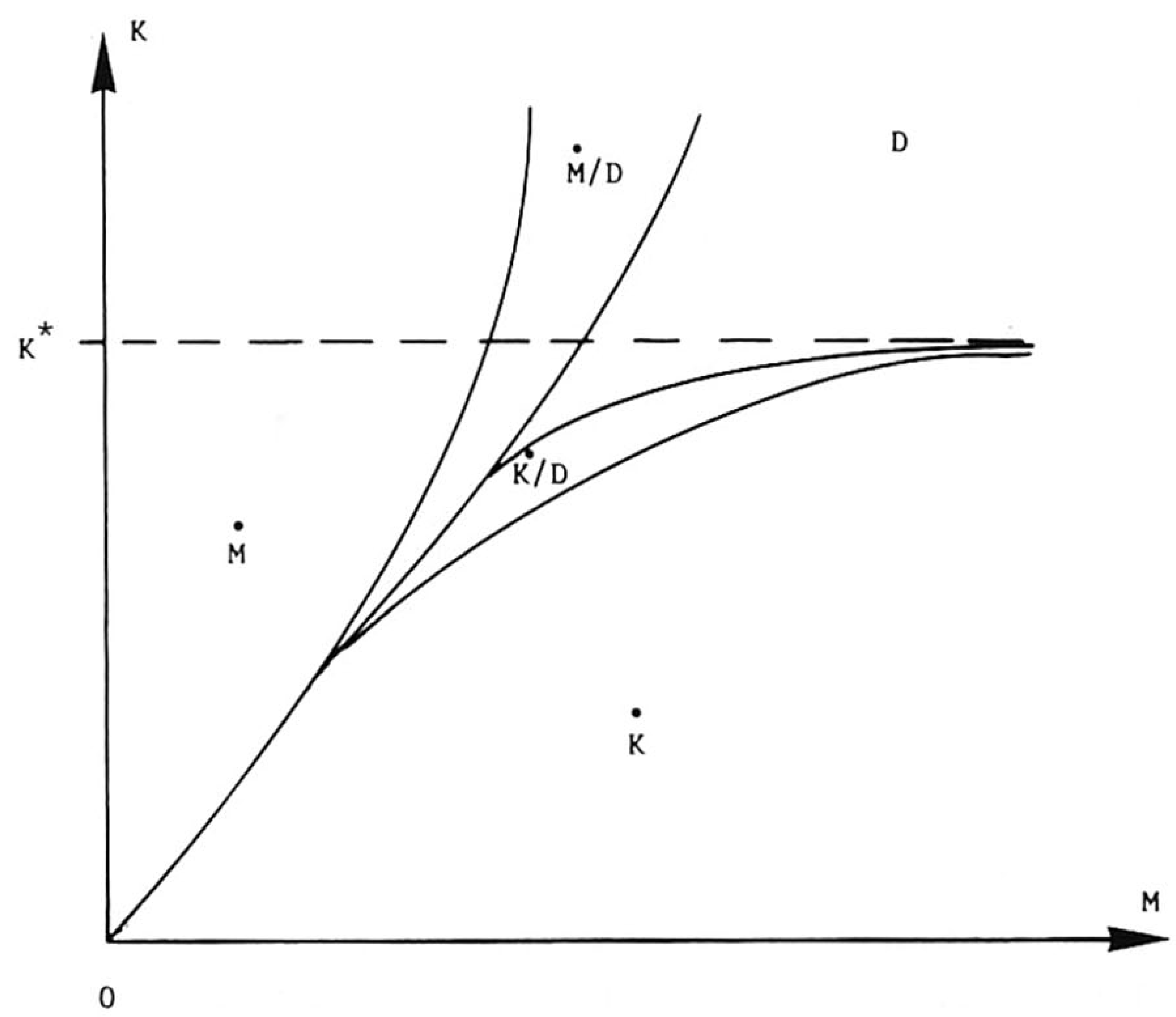

Figure 2. The optimal solution of the model under the assumptions (a), (b) and (c)

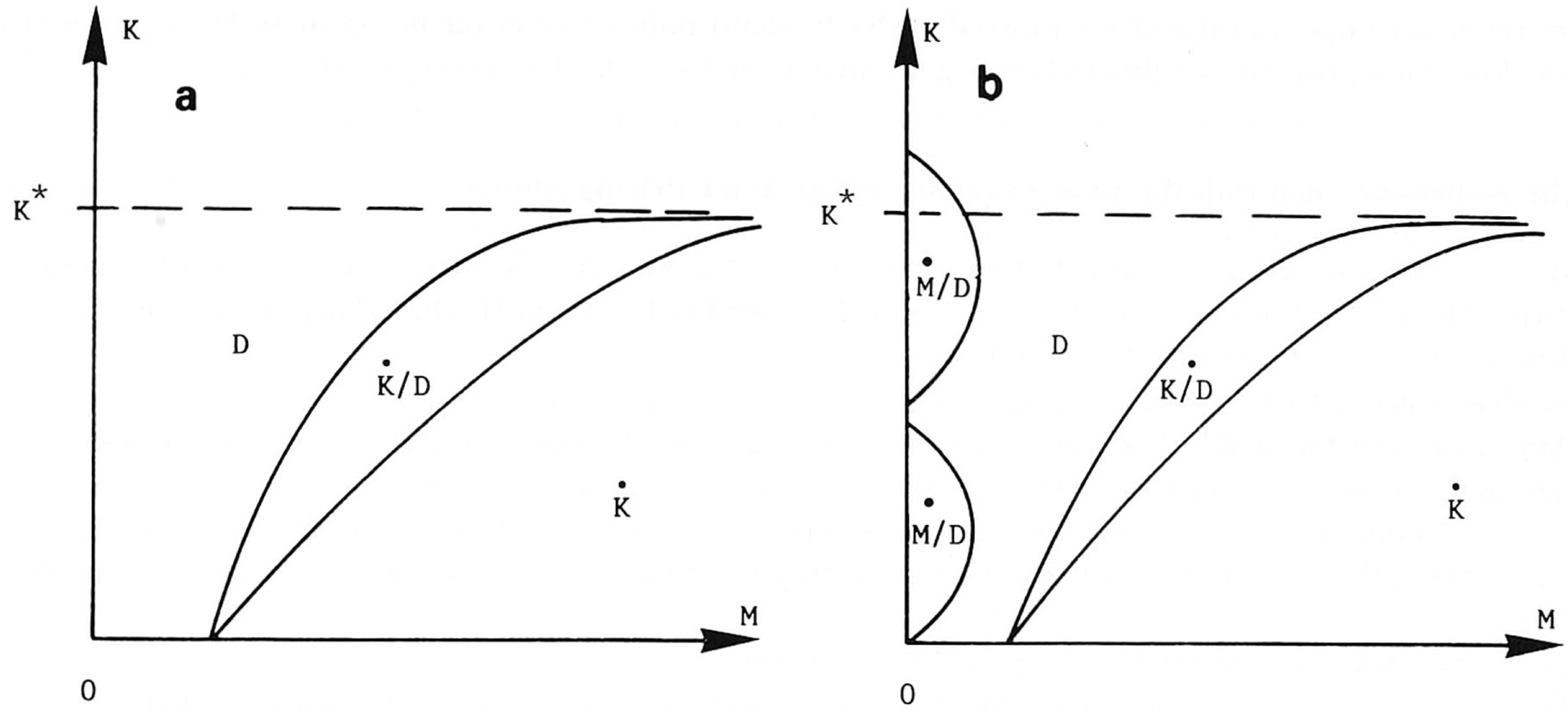

Figure 3. The optimal solution of the model under the assumptions (a) and (c)

it holds that $D=0$, and on the boundary with the dividend-region (D), D is equal to $S(K)$. In between dividend is such that $\partial U/\partial M = \mathrm{d}V/\mathrm{d}D$, so the increase of the value of the firm due to one extra unit of cash is equal to the marginal utility of dividend.

Concerning the investment/dividend-region, on its boundary with the investment-region ($\dot{K}$) it holds that $\dot{K}=S(K)$ and $D=0$, and on the boundary with the dividend-region $\dot{K}=0$ and $D=S(K)$. In the rest of this region $\dot{K}$ and D are such that $\partial U/\partial K = \partial V/\partial D$.

If we drop assumption (b), the solutions presented in Figure 3 can emerge. In Figure 3(a), the shareholders do not want the firm to increase the amount of cash, even if cash is almost zero. An economic reason could be that the firm has to cope with a very risky environment. Because of the bankruptcy risk the shareholders want to obtain dividend as soon as possible. They do not want to increase the cash balance first, because there is a risk of the firm going bankrupt before the dividend payout starts. Of course, this solution will only be optimal in very extreme situations such as under severe threats of war, revolution, a sudden decrease of the market, etc. Concerning Figure 3 we state the following proposition, from which the proof can be obtained from the author upon request.

Proposition 3. *A necessary condition for Figure* 3(a) *to be optimal is that, for all* K, *it holds that*:

$$\left(V(D)/\frac{\mathrm{d}V}{\mathrm{d}D}\cdot D\right)\Bigg|_{D=S(K)} \leqslant \frac{\sigma\sqrt{i}}{\sqrt{2}}. \tag{20}$$

Concerning Figure 3(b), *the cash/dividend-region includes the* K*-axis for those* K *which satisfy the following expression*:

$$\left(V(D)/\frac{\mathrm{d}V}{\mathrm{d}D}\cdot D\right)\Bigg|_{D=S(K)} > \frac{\sigma\sqrt{i}}{\sqrt{2}}. \quad \square \tag{21}$$

The result of Proposition 3 can be nicely interpreted from an economical point of view. First, notice that $[V(D)/(\mathrm{d}V/\mathrm{d}D)\cdot D]=1/e$, where e represents the elasticity of utility (V) with respect to dividends (D). A large elasticity implies a relatively high increase of utility due to an increase of dividend. So, if the elasticity is large for $D=S(K)$, the shareholders prefer a dividend policy instead of a mixed cash/dividend policy. A shareholder with a large time preference rate wants to obtain a large amount of dividends as soon as possible and if investment is very risky, shareholders want to obtain dividends immediately because of the high risk of bankruptcy. If expression (20) holds, i, σ and the elasticity of V with respect to

D are relatively large and therefore a mixed cash/dividend policy will never be optimal. If the amount of cash is low, the optimality of this policy is guaranteed for those K that satisfy relation (21).

4. The model extended with the Intertemporal Capital Asset Pricing Model

In this section we connect the Intertemporal Capital Asset Pricing Model with our dynamic model of the firm. The ICAPM was invented by Merton [11,12] and is based on the following assumptions, which are briefly stated here (see also [5, p. 604]):

- the shareholders face a perfect capital market;
  (Application of the ICAPM requires that investors face a perfect capital market. In our framework the shareholders are the investors, who consider the firm as a project in which they have invested their money. Analogous to an example of Constantinides, we assume that the firm may take no action at times within the time interval $(0, T)$ such as issuing or buying back equity and issuing debt (see [5, p. 609]).)
- the discrete-time rate of return is lognormally distributed;
  (This is equivalent to saying that the continuous-time rate of return is normally distributed. Our problem satisfies this assumption, as can be obtained from Equation (24) below.)
- investors have homogeneous expectations;
- investors are risk-averse.

Under these assumptions the following ICAPM-relation becomes applicable:

$$\alpha_{\mathrm{F}} - r = \lambda \sigma_{\mathrm{Fm}} / \sigma_{\mathrm{m}}, \tag{22}$$

in which

α_{F} = the firm's expected rate of return per unit time,
r = riskless borrowing–lending rate,
σ_{m} = standard deviation of the rate of return per unit time of the market portfolio,
σ_{Fm} = covariance between the rates of return per unit time of the firm and the market portfolio,
λ = $(\alpha_{\mathrm{m}} - r)/\sigma_{\mathrm{m}}$ = market price per unit risk, where α_{m} = expected rate of return per unit time of the market portfolio,

Following a method described by Constantinides [5], we derive α_{F} and σ_{Fm} for our model. First, we apply Itô's lemma (see [10, p. 89]) to the value function

$$\begin{aligned} \mathrm{d}U &= U(M + \mathrm{d}M,\ K + \mathrm{d}K) - U(M,\ K) \\ &= \left(\frac{\partial U}{\partial M}(S(K) - \dot{K} - D) + \frac{\partial U}{\partial K}\dot{K} + \tfrac{1}{2}\sigma^2 S^2(K)\frac{\partial^2 U}{\partial M^2} \right) \mathrm{d}t + \frac{\partial U}{\partial M}\sigma S(K)\,\mathrm{d}B. \end{aligned} \tag{23}$$

After assuming that the firm is optimally controlled and using (23), we can state the firm's rate of return:

$$\begin{aligned} &\frac{U(M + \mathrm{d}M,\ K + \mathrm{d}K) + V(D)\,\mathrm{d}t - U(M,\ K)}{U(M,\ K)} \\ &\quad = \frac{1}{U(M,\ K)}\left(V(D) + \frac{\partial U}{\partial M}(S(K) - \dot{K} - D) + \frac{\partial U}{\partial K}\dot{K} + \tfrac{1}{2}\sigma^2 S^2(K)\frac{\partial^2 U}{\partial M^2} \right) \mathrm{d}t \\ &\qquad + \frac{\frac{\partial U}{\partial M}\sigma S(K)}{U(M,\ K)}\mathrm{d}B. \end{aligned} \tag{24}$$

From (13) and (24) we derive

$$\alpha_{\mathrm{F}} = i, \tag{25}$$

$$\sigma_{\mathrm{Fm}} = \rho_{\mathrm{Fm}}\sigma_{\mathrm{m}}\sigma S(K)\left(\frac{\partial U}{\partial M}\right)/U \tag{26}$$

in which

ρ_{Fm} = instantaneous correlation coefficient between the firm's return and the market return.

After substituting (25) and (26) in (22) we get the expression (27) for the time preference rate of the shareholders. This result has not been derived, to the best of our knowledge, in the literature until now:

$$i = r + \lambda \rho_{\mathrm{Fm}} \sigma S(K)\left(\frac{\partial U}{\partial M}\right)/U. \tag{27}$$

So, analogous to a result of the static CAPM, the shareholders' time preference rate consists of the sum of the riskless market interest rate and a risk premium. The risk premium depends on the market price per unit risk (λ), the correlation coëfficient between the returns of the firm and the market (ρ_{Fm}), the standard deviation of the earnings function ($\sigma S(K)$) and the sensitivity of the value of the firm with respect to a marginal change in the amount of cash ($(\partial U/\partial M)/U$). This makes sense, because the amount of risk of the return of the firm's investment depends on the amount of cash. If the latter is low there is a high risk of bankruptcy, and the influence of a marginal change in the cash balance on the value of the firm is high. If the firm has a large amount of cash, then one unit increase or decrease of this amount only has a minor influence on the expected utility stream of dividends. Thus a marginal change in the cash position has a minor influence on the value of the firm. We conclude that the risk premium will be high if the amount of cash is low.

We now state the following proposition:

Proposition 4. *If M goes to infinity the boundaries between K/D and D and between $\dot{K}$ and $\dot{K}/D$ approach an asymptote which is situated on the level $\hat{K}$, determined by* $(\mathrm{d}S/\mathrm{d}K)|_{K=\hat{K}} = r$.

Proof. This proof follows the same steps as the proof of Proposition 1 and is therefore omitted. □

In the dynamic model of the firm, in which the CAPM is not incorporated, the asymptote corresponds to a level of K, determined by $\mathrm{d}S/\mathrm{d}K = i$. The reason for the difference between this result and that of Proposition 4 is that now the shareholders' time preference rate is equal to r if M has an infinite value. Due to the fact that there is no risk of bankruptcy, then $\partial U/\partial M$ is equal to zero and (27) shows that the shareholders' time preference rate equals the riskless market interest rate. To conclude briefly: for M sufficiently large, risk tends to disappear.

5. Summary

In this paper the analysis of deterministic dynamic models of the firm is extended by incorporating a stochastic component in the earnings function. Due to this extension earnings may fall below the expenses level and the firm needs cash to meet its obligations during those periods. Our starting point is the pathbreaking work of Bensoussan and Lesourne [3] who analysed a stochastic model with dividend maximization. Using dynamic programming they proved that—depending on the amount of capital goods, the amount of cash, the shareholders' time preference rate and the variance of the earnings—it is optimal for the firm to choose one of the following three ways of spending its expected earnings: increase the amount of cash, invest the money or pay it out as dividend.

We extend the Bensoussan and Lesourne model by changing the objective from dividend maximization into maximizing a concave utility function of dividends, implying risk-averse investor behavior. The model is connected with a dynamic version of the Capital Asset Pricing Model invented by Merton [11,12]. Using a method by Constantinides [5], we were able to derive a new formula for the shareholders' time preference rate, which consists of the riskless interest rate and a risk premium. We further demonstrate that, in contrast with the Bensoussan and Lesourne model, also a mixed investment dividend policy and a mixed cash dividend policy could be optimal. Another interesting result is that the shareholders' time preference rate is equal to the riskless market interest rate if the firm possesses a large amount of cash, thus having no bankruptcy risk.

Appendix

We first derive some relations which hold for the different policies.

Investment policy
After substituting (15) in (13) we get:

$$iU = S(K)\frac{\partial U}{\partial K} + S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{A.1}$$

Cash policy
After substituting (16) in (13) we get:

$$iU = S(K)\frac{\partial U}{\partial M} + S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{A.2}$$

This differential equation can be solved:

$$U = k_1(K)\exp\left[r_1\frac{M}{S(K)}\right] + k_2(K)\exp\left[r_2\frac{M}{S(K)}\right], \tag{A.3}$$

in which

$k_1(K)$ and $k_2(K)$ are unknown functions,

$$r_1 = \left[-1 + (1 + 2\sigma^2 i)^{1/2}\right]/\sigma^2, \tag{A.4}$$

$$r_2 = \left[-1 - (1 + 2\sigma^2 i)^{1/2}\right]/\sigma^2. \tag{A.5}$$

Dividend policy
After substituting (17) in (13) we get

$$iU = V(D)\,|_{D=S(K)} + S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{A.6}$$

This differential equation can be solved:

$$U = \frac{V(D)\,|_{D=S(K)}}{i} + c_1(K)\exp\left(\frac{M\sqrt{2i}}{\sigma S(K)}\right) + c_2(K)\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right), \tag{A.7}$$

in which

$c_1(K)$ and $c_2(K)$ are unknown functions.

Cash / dividend policy
After substituting (18) in (13) we get

$$iU = V(D) + (S(K) - D)\frac{\partial U}{\partial M} + S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{A.8}$$

Investment / dividend policy

After substituting (19) in (13) we get

$$iU = V(D) + (S(K) - D)\frac{\partial U}{\partial K} + S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{A.9}$$

A.1. Derivation of the optimal policies

Equation (13) can be rewritten as follows:

$$iU = \max_{\substack{\dot{K},\, D \geqslant 0 \\ \dot{K} + D \leqslant S(K)}} \left\{ V(D) - D\frac{\partial U}{\partial M} + \dot{K}\left(\frac{\partial U}{\partial K} - \frac{\partial U}{\partial M}\right)\right\} + S(K)\frac{\partial U}{\partial M} + \tfrac{1}{2}\sigma^2 S^2(K)\frac{\partial^2 U}{\partial M^2}. \tag{A.10}$$

To maximize the part in brackets, subject to the restrictions, we formulate the Lagrangian:

$$L = V(D) - D\frac{\partial U}{\partial M} + \dot{K}\left(\frac{\partial U}{\partial K} - \frac{\partial U}{\partial M}\right) + \lambda_1\dot{K} + \lambda_2 D + \lambda_3(S(K) - \dot{K} - D). \tag{A.11}$$

From (A.11) we can derive the following optimality conditions:

$$\frac{\partial L}{\partial D} = \frac{dV}{dD} - \frac{\partial U}{\partial M} + \lambda_2 - \lambda_3 = 0, \tag{A.12}$$

$$\frac{\partial L}{\partial \dot{K}} = \frac{\partial U}{\partial K} - \frac{\partial U}{\partial M} + \lambda_1 - \lambda_3 = 0, \tag{A.13}$$

$$\lambda_1 \dot{K} = 0, \tag{A.14}$$

$$\lambda_2 D = 0, \tag{A.15}$$

$$\lambda_3(S(K) - \dot{K} - D) = 0. \tag{A.16}$$

Now, it is not difficult anymore to obtain that, due to (A.12) through (A.16) and taking into account (5), the following policies are optimal depending on the relative size of $\partial U/\partial M$, $\partial U/\partial K$ and dV/dD for different values of D.

Investment policy: $dM = \sigma S(K)\,dB$, $D = 0$, $dK = S(K)\,dt$;
optimal if:

$$\frac{\partial U}{\partial K} \geqslant \max\left(\left.\frac{dV}{dD}\right|_{D=0}, \frac{\partial U}{\partial M}\right).$$

Cash policy: $dM = S(K)\,dt + \sigma S(K)\,dB$, $D = 0$, $dK = 0$;
optimal if:

$$\frac{\partial U}{\partial M} \geqslant \max\left(\left.\frac{dV}{dD}\right|_{D=0}, \frac{\partial U}{\partial K}\right).$$

Dividend policy: $dM = \sigma S(K)\,dB$, $D = S(K)$, $dK = 0$;
optimal if:

$$\left.\frac{dV}{dD}\right|_{D=S(K)} \geqslant \max\left(\frac{\partial U}{\partial K}, \frac{\partial U}{\partial M}\right).$$

Cash / dividend policy: $dM = (S(K) - D)\,dt + \sigma S(K)\,dB$, $D \geqslant 0$, $dK = 0$;
optimal if:

$$\frac{dV}{dD} = \frac{\partial U}{\partial M} \geqslant \frac{\partial U}{\partial K}.$$

Investment / dividend policy: $\mathrm{d}M = \sigma S(K)\,\mathrm{d}B,\ D \geqslant 0,\ \mathrm{d}K = (S(K) - D)\,\mathrm{d}t$; optimal if:

$$\frac{\mathrm{d}V}{\mathrm{d}D} = \frac{\partial U}{\partial K} \geqslant \frac{\partial U}{\partial M}.$$

A.2. Proof of Proposition 1

Due to (16) and (18) we can derive that at the boundary between the cash- and the cash/dividend-region it holds that:

$$\frac{\partial U}{\partial M} = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{A.17}$$

Because the cash-region includes the K-axis (see assumption (b) in Section 3), we can derive due to (14) and (A.3):

$$k_1(K) = -k_2(K) \tag{A.18}$$

and (A.3), (A.17) and (A.18) yield the following expression for the boundary:

$$k_1(K)\left(\frac{r_1}{S(K)}\exp\left(\frac{r_1 M}{S(K)}\right) - \frac{r_2}{S(K)}\exp\left(\frac{r_2 M}{S(K)}\right)\right) = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{A.19}$$

Due to (A.4) we know that r_1 is positive and from (A.19) we can now derive that if M goes to infinity then also K goes to infinity, if $k_1(K) \neq 0$. $k_1(K)$ is not zero, because if it is zero, due to (A.3) and (A.18) U would be zero for this K and every $M > 0$. This makes no sense, for U represents the value of the firm.

Now, we turn to the boundary between the cash/dividend- and the dividend-region.

Concerning the dividend-region: due to the fact that U must be finite and the assumption that the dividend-region exists for finite K and infinite M (this seems reasonable from an economical point of view), we can derive from (A.7):

$$c_1(K) = 0. \tag{A.20}$$

Due to (17) and (18) we get that at the boundary between cash/dividend and dividend it holds that

$$\frac{\partial U}{\partial M} = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)}. \tag{A.21}$$

From (A.7), (A.20) and (A.21) we can derive the following expression for the boundary:

$$\frac{-c_2(K)\sqrt{2i}}{\sigma S(K)}\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right) = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)}. \tag{A.22}$$

If $c_2(K)$ is larger than minus infinity (of course c_2 must be negative), K must be infinite if M is infinite, because otherwise (A.22) could not exist. $c_2(K)$is unequal to minus infinity, because otherwise due to (A.7) and (A.20) U would become negative for this particular K and finite M, which is excluded by (12).

Appendix B

In this appendix we show which conditions must be fulfilled, such that the assumptions (a), (b) and (c) of Section 3 are satisfied. We start the analysis by assuming that the cash-region covers the K-axis and then we proceed by treating the cash–cash/dividend boundary, the cash/dividend–dividend boundary, the dividend–investment/dividend boundary and the investment/dividend–investment boundary, respectively.

Assumption. The cash-region covers the K-axis.

This assumption coincides with assumption (b), that states: The firm keeps its cash if K is high enough but the cash situation is poor.

From Appendix A we obtain that a cash policy is optimal if:

$$\frac{\partial U}{\partial M} \geqslant \max\left[\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}, \frac{\partial U}{\partial K}\right]. \tag{B.1}$$

Because the cash-region covers the K-axis, (14) and (A.3) must coincide and therefore it holds that

$$k_1(K) = -k_2(K). \tag{B.2}$$

So, (A.3) can be written as

$$U = k_1(K)\left[\exp\left(\frac{r_1 M}{S(K)}\right) - \exp\left(\frac{r_2 M}{S(K)}\right)\right]. \tag{B.3}$$

U represents the value of the firm and is always positive. Therefore $k_1(K) > 0$ for all K.

Now we can obtain:

$$\left.\frac{\partial U}{\partial K}\right|_{M=0} = 0 \quad \text{because of (14)}, \tag{B.4}$$

$$\left.\frac{\partial U}{\partial M}\right|_{M=0} = \frac{k_1(K)}{S(K)}(r_1 - r_2) > 0. \tag{B.5}$$

Now, (B.1), (B.4) and (B.5) lead to the following necessary condition for the optimality of the cash-region covering the K-axis.

$$\frac{k_1(K)}{S(K)}(r_1 - r_2) > \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{B.6}$$

Because $k_1(K) > 0$ it should not be difficult to find functions $S(K)$ and $V(D)$ such, that condition (B.6) is satisfied.

The cash–cash / dividend boundary

From (A.3), (A.17) and (A.18) we obtain the following expression for this boundary:

$$\frac{\partial U}{\partial M} = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0} \Rightarrow k_1(K)\left[\frac{r_1}{S(K)}\exp\left(\frac{r_1 M}{S(K)}\right) - \frac{r_2}{S(K)}\exp\left(\frac{r_2 M}{S(K)}\right)\right] = \left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{B.7}$$

Combining (B.6) and (B.7) implies

$$\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}\left[r_1 \exp\left(\frac{r_1 M}{S(K)}\right) - r_2 \exp\left(\frac{r_2 M}{S(K)}\right)\right] < (r_1 - r_2)\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0},$$

so:

$$r_1 \exp\left(\frac{r_1 M}{S(K)}\right) - r_1 < r_2 \exp\left(\frac{r_2 M}{S(K)}\right) - r_2. \tag{B.8}$$

Now, we can conclude that under condition (B.6), expression (B.8) is a necessary condition for the existence of the boundary between the cash-region and the cash/dividend-region, which implies that on this boundary M and K are such, that $M/S(K)$ is not too large.

The cash-dividend–dividend boundary

Equations (A.7), (A.20) and (A.21) yield the following expression for this boundary:

$$\frac{\partial U}{\partial M}=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)} \Rightarrow \frac{-c_2(K)\sqrt{2i}}{\sigma S(K)}\exp\left[\frac{-M\sqrt{2i}}{\sigma S(K)}\right]=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)}. \tag{B.9}$$

As stated in Appendix A, $c_2(K)$ is negative. From (B.9) we derive:

$$\frac{\partial^2 U}{\partial M^2}=\frac{2ic_2(K)}{\sigma^2 S^2(K)}\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right)<0 \quad \text{because } c_2(K)<0. \tag{B.10}$$

For the dividend policy to be optimal, it must hold that (see Appendix A)

$$\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)} \geqslant \max\left(\frac{\partial U}{\partial K},\ \frac{\partial U}{\partial M}\right). \tag{B.11}$$

From (B.9)–(B.11) we can derive that the dividend-region is situated at the right-hand side of the cash/dividend-region. So, we can conclude that the dividend-region exists for M sufficiently high, which coincides with assumption (a).

The dividend–investment / dividend boundary

Due to (17), (19), (A.7) and (A.20) we can derive the following expression for this boundary:

$$\frac{\partial U}{\partial K}=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)} \Rightarrow$$

$$\frac{\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)}\frac{\mathrm{d}S}{\mathrm{d}K}}{i}+\left[\frac{\mathrm{d}c_2}{\mathrm{d}K}+\frac{c_2(K)M\sqrt{2i}\,(\mathrm{d}S/\mathrm{d}K)}{\sigma S^2(K)}\right]\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right)=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=S(K)}. \tag{B.12}$$

$c_2(K)$ is finite, because otherwise U would be infinite (cf. (A.7) and (A.20)) and we can obtain from (12) that U must have a finite value. From the assumption that $\mathrm{d}U/\mathrm{d}K$ exists (see below (12)) we can obtain that also $\mathrm{d}c_2/\mathrm{d}K$ must be finite (cf. (A.7) and (A.20)).

Because $c_2(K)$ and $\mathrm{d}c_2/\mathrm{d}K$ have finite values, the second term of $\partial U/\partial K$ tends to disappear for increasing M. So, if we want to analyze (B.12) for sufficiently large values of M, we only need to evaluate the first term of $\partial U/\partial K$. Because a dividend policy is only optimal if (B.11) holds, we derive from (B.12) that in the dividend-region K must exceed K^*, where $(\mathrm{d}S/\mathrm{d}K)|_{K=K^*}=i$ (notice that S is concave).

Hence, we can conclude that the firm distributes dividends if M and K are sufficiently high, which coincides with assumption (a).

The investment / dividend–investment boundary

From (15) and (19) we get that at the boundary between investment and investment/dividend it holds that

$$\frac{\partial U}{\partial K}=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{B.13}$$

So, D is equal to zero at this boundary. From (A.9) and (B.13) we derive:

$$iU=S(K)\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}+S^2(K)\tfrac{1}{2}\sigma^2\frac{\partial^2 U}{\partial M^2}. \tag{B.14}$$

The solution of this differential equation is equal to

$$U=\frac{(\mathrm{d}V/\mathrm{d}D)|_{D=0}S(K)}{i}+c_3(K)\exp\left(\frac{M\sqrt{2i}}{\sigma S(K)}\right)+c_4(K)\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right), \tag{B.15}$$

in which

$c_3(K)$ and $c_4(K)$ are unknown functions.

Assuming that the boundary exists for infinite M and finite K, which seems reasonable since it is likely that this boundary is situated below the boundary between dividend and investment/dividend, c_3 must be equal to zero, because U must have a finite value. Notice that equation (B.15) only holds in the investment/dividend-region for $D=0$ and not in the whole region. Due to (B.13), (B.15) and $c_3(K)=0$ we can derive the following expression for the boundary:

$$\frac{\mathrm{d}U}{\mathrm{d}K}=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0} \Rightarrow$$

$$\frac{(\mathrm{d}V/\mathrm{d}D)\,|_{D=0}(\mathrm{d}S/\mathrm{d}K)}{i}+\left(\frac{\mathrm{d}c_4}{\mathrm{d}K}+\frac{c_4(K)M\sqrt{2i}\,(\mathrm{d}S/\mathrm{d}K)}{\sigma S^2(K)}\right)\exp\left(\frac{-M\sqrt{2i}}{\sigma S(K)}\right)=\left.\frac{\mathrm{d}V}{\mathrm{d}D}\right|_{D=0}. \tag{B.16}$$

Because $\mathrm{d}c_4/\mathrm{d}K$ and $c_4(K)$ do not have an infinite value (this can be obtained from the facts that U and $\mathrm{d}U/\mathrm{d}K$ are finite), the second term of $\partial U/\partial K$ disappears if M is sufficiently high. An investment policy can only be optimal if $\partial U/\partial K \geqslant (\mathrm{d}V/\mathrm{d}D)\,|_{D=0}$ (cf. (15)). Now, we can derive from (B.16), that in the investment-region K must be less than K^* $((\mathrm{d}S/\mathrm{d}K)\,|_{K=K^*}=i)$ for M sufficiently high. So the firm invests if the risk of bankruptcy is limited (M sufficiently high), while the amount of equipment is low ($K<K^*$). This statement can be found under assumption (c).

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